

Optimal Source Coding with Signal Transfer Function Constraints

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I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

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Abstract

This thesis presents results on optimal coding and decoding of discrete-time stochastic signals, in the sense of minimizing a distortion metric subject to a constraint on the bit-rate and on the signal transfer function from source to reconstruction.

The first (preliminary) contribution of this thesis is the introduction of new distortion metric that extends the *mean squared error* (MSE) criterion. We give this extension the name *Weighted-Correlation MSE* (WCMSE), and use it as the distortion metric throughout the thesis. The WCMSE is a *weighted* sum of two components of the MSE: the variance of the error component uncorrelated to the source, on the one hand, and the remainder of the MSE, on the other. The WCMSE can take account of signal transfer function constraints by assigning a larger weight to deviations from a target signal transfer function than to source-uncorrelated distortion.

Within this framework, the second contribution is the solution of a family of feedback quantizer design problems for wide sense stationary sources using an additive noise model for quantization errors. These associated problems consist of finding the frequency response of the filters deployed around a scalar quantizer that minimize the WCMSE for a fixed quantizer *signal-to-(granular)-noise ratio* (SNR). This general structure, which incorporates pre-, post-, and feedback filters, includes as special cases well known source coding schemes such as *pulse coded modulation* (PCM), *Differential Pulse-Coded Modulation* (DPCM), Sigma Delta ($\Sigma\Delta$) converters, and noise-shaping coders. The optimal frequency response of each of the filters in this architecture is found for each possible subset of the remaining filters being given and fixed. These results are then applied to oversampled feedback quantization. In particular, it is shown that, within the linear model used, and for a fixed quantizer SNR, the MSE decays exponentially with oversampling ratio, provided optimal filters are used at each oversampling ratio. If a subtractively dithered quantizer is utilized, then the noise model is exact, and the SNR constraint can be directly related to the bit-rate if entropy coding is used, regardless of the number of quantization levels. On the other hand, in the case of fixed-rate quantization, the SNR is related to the number of quantization levels, and hence to the bit-rate, when overload errors are negligible. It is shown that, for sources with

unbounded support, the latter condition is violated for sufficiently large oversampling ratios. By deriving an upper bound on the contribution of overload errors to the total WCMSE, a lower bound for the decay rate of the WCMSE as a function of the oversampling ratio is found for fixed-rate quantization of sources with finite or infinite support.

The third main contribution of the thesis is the introduction of the *rate-distortion function* (RDF) when WCMSE is the distortion metric, denoted by WCMSE-RDF. We provide a complete characterization for Gaussian sources. The resulting WCMSE-RDF yields, as special cases, Shannon's RDF, as well as the recently introduced *RDF for source-uncorrelated distortions* (RDF-SUD). For cases where only source-uncorrelated distortion is allowed, the RDF-SUD is extended to include the possibility of linear-time invariant feedback between reconstructed signal and coder input. It is also shown that feedback quantization schemes can achieve a bit-rate only 0.254 bits/sample above this RDF by using the same filters that minimize the reconstruction MSE for a quantizer-SNR constraint.

The fourth main contribution of this thesis is to provide a set of conditions under which knowledge of a realization of the RDF can be used directly to solve encoder-decoder design optimization problems. This result has direct implications in the design of subband coders with feedback, as well as in the design of encoder-decoder pairs for applications such as networked control.

As the fifth main contribution of this thesis, the RDF-SUD is utilized to show that, for Gaussian stationary sources with memory and MSE distortion criterion, an upper bound on the information-theoretic causal RDF can be obtained by means of an iterative numerical procedure, at all rates. This bound is tighter than 0.5 bits/sample. Moreover, if there exists a realization of the causal RDF in which the reconstruction error is jointly stationary with the source, then the bound obtained coincides with the causal RDF. The iterative procedure proposed here to obtain $\overline{R_c^{it}}(D)$ also yields a characterization of the filters in a scalar feedback quantizer having an operational rate that exceeds the bound by less than 0.254 bits/sample. This constitutes an upper bound on the optimal performance theoretically attainable by any causal source coder for stationary Gaussian sources under the MSE distortion criterion.

Contents

1	Introduction	7
1.1	Background and Motivation	7
1.1.1	Distortion Metrics	9
1.1.2	Signal Transfer Function Constraints	10
1.1.3	Architectural Limitations	12
1.1.4	Quantization and Entropy Coding Constraints	13
1.1.5	Delay	14
1.2	Previous Related Work	16
1.2.1	MSE Extensions	16
1.2.2	Brief Review of Source Coding Paradigms	16
1.2.3	Related Existing Results on Causal and Delay-Free Source Coding	23
1.3	Overview of the Main Contributions	24
1.3.1	A Two-Parameter Frequency-Weighted MSE	24
1.3.2	WCMSE Optimal Frequency Responses for Scalar Feedback Quantizers	26
1.3.3	The RDF for Gaussian Sources with WCMSE as Fidelity Criterion	27
1.3.4	Using Realizations of the RDF to Design Optimal Source Coders	27
1.3.5	Results on the Causal Quadratic Gaussian Rate-Distortion Function	28
1.3.6	Summary of the Main Contributions	28
1.4	Associated Publications	29
2	Preliminaries	31
2.1	Notation	31
2.2	Definitions	33
2.3	Basic Information Theoretical Concepts and Results	35
2.3.1	Entropy	35

2.3.2	Mutual Information	38
2.4	Scalar Memoryless Quantization	40
2.4.1	Uniform Scalar Quantization	40
2.4.2	Subtractively Dithered Uniform Scalar Quantization	41
3	WCMSE-Optimal Filters for a Given Quantizer SNR	43
3.1	Introduction	43
3.2	Analysis Model and Assumptions	46
3.2.1	Feedback Quantizer Equations	46
3.2.2	Assumptions	48
3.2.3	Optimization Constraints	51
3.2.4	Analysis Model	52
3.3	$F(z)$ and $A(z)$ (or $B(z)$) Given	54
3.3.1	$F(z)$ and $A(z)$ Given	54
3.3.2	$F(z)$ and $B(z)$ Given	55
3.4	$F(z)$ and the Signal Transfer Function Given	56
3.5	$F(z)$ Given	59
3.6	$A(z)$ and $B(z)$ Given	63
3.7	$B(z)$ Given	72
3.8	$H(z)$ (pre-filter) Given	73
3.9	No Constraints:	
The WCMSE Optimal Feedback Quantizer		74
3.9.1	Special Cases	78
3.9.2	The Importance of Taking Account of Fed Back Quantization Noise	80
3.10	Comparative Analysis	83
3.10.1	Optimal Frequency Responses	83
3.10.2	Optimal Signal Spectra	83
3.10.3	Optimal Performance	86
3.11	Simulation Example	86
3.11.1	Simulation Setup	86
3.11.2	Results	89
3.12	Oversampled Feedback Quantization	92
3.12.1	Introduction	92
3.12.2	The Oversampled Case Without Clipping/Overload	93

3.12.3	The Oversampled Case With Clipping	96
3.13	Summary	104
3.14	Appendix	104
4	The WCMSE-Rate-Distortion Function	107
4.1	Introduction	107
4.2	Preliminaries	108
4.2.1	The WCMSE is Not Linear in the PDF of Source and Reconstruction	108
4.2.2	The Reconstruction Error Must Be Jointly Gaussian with the Source	110
4.3	WCMSE-RDF for Gaussian Scalar Sources	112
4.3.1	Geometrical Interpretation	115
4.3.2	Convexity of $R_{a,b}(D)$	117
4.4	WCMSE RDF For Gaussian Vector Sources	119
4.4.1	Preliminary Results	119
4.4.2	$R_{a,b}(D)$ for Gaussian Vectors	125
4.5	WCMSE RDF For Gaussian Stationary Processes	130
4.5.1	Distortion Spectra	135
4.5.2	Special Cases	136
4.6	WCMSE-RDF For Vector Processes	139
4.7	Image Processing Example	143
4.8	Achievability	144
4.8.1	Background on Dithered Lattice Quantization	147
4.8.2	Achievability of $R_{a,b}(D)$	148
4.9	$R^\perp(D)$ Within Feedback Loops	151
4.9.1	The Directed Version of $R^\perp(D)$	152
4.9.2	The Gaussian Case	154
4.10	Summary	162
4.11	Appendix	163
5	Using Realizations of the RDF to Design Optimal Source Coders	165
5.1	Introduction	165
5.2	Conditions for Scalar Processes	167
5.2.1	All Three Degrees of Freedom are Necessary	172
5.2.2	Entropy Coding with Memory is an Extra Degree of Freedom	172

5.3	Conditions for Vector Sources	176
5.4	Conditions for Vector Processes	184
5.5	Summary	192
6	Bounds on the Causal RDF for Gaussian Processes	193
6.1	Introduction	193
6.2	Obtaining the Stationary Causal RDF	197
6.3	Upper Bound on the Operational Causal RDF	205
6.4	Summary	206
6.5	Appendix	206
7	Conclusions	209
7.1	Overview	209
7.2	Main Contributions	209
7.3	Directions for Future Research	212